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The growth simulation of circular buckling-driven delamination

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Abstract

Some closed-form equations for the coupling problem of buckling and growth of circular delamination are derived by recourse to the moving boundary variational principle. The axisymmetric buckling of a circular delamination subjected to an equal bi-axial compression is analysed by using high-order perturbation expansion. The axisymmetric buckled delamination has the following properties: under a certain residual pressure, there exist two characteristic radii, namely the critical radius R_c and growing radius R_g; for a certain interface toughness, the blister has three configuration of stationary, stable growth and unstable growth with increasing the loads. Under a higher edge thrust, the nonaxisymmetric secondary buckling will occur on the base of axisymmetric buckling and then the toughness and the driving force of the interface crack will be different along the delamination front. So the growth of circular delamination will not be self-similar. Without any assumption regarding the delamination front, the configurations of the blister with several nonaxisymmetric buckling modes $n = 2, 3, 6, 8$ are simulated. The nonaxisymmetric growth process for the nonaxisymmetric buckling mode $n = 2$ is simulated also under a sequence of loads. © 1998 Elsevier Science Ltd. All rights reserved.

1. Introduction

The compressed film in the film/substrate system usually buckles away from the substrate because of the existence of delamination. The buckling induces stress concentration along the delamination front and leads to further growth of the delamination. This failure phenomenon usually exists in laminate composite, microelectronic element, microelectronic packaging and microelectro–mechanical system (Yin, 1985; Argon et al., 1988, 1989; Hutchinson et al., 1992). Initially, small flaws tend to have smooth, nearly circular, boundaries and to exhibit axisymmetric deflections. Under this condition, the driving force (i.e., the energy release rate) along the delamination front distributes uniformly, so is the toughness of the interface crack, which is related to

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the mixity of interface crack. So the growth of the delamination is axisymmetric if the driving force is greater than the toughness of the interface crack along the periphery of the blister. As the load increases, or as the flaw grows in size, the film is observed to fold and the boundary to become wavy (Hutchinson et al., 1992; Jensen and Thouless, 1993; Ortiz and Gioia, 1994; Nilsson and Giannakopoulos, 1995; Zhang and Yu, 1996a,b). The reason of nonaxisymmetric growth of buckling-driven delamination may be the instability of the boundary, or the transition of the buckling mode. On the base of axisymmetric deformation, Hutchinson et al. (1992) and Jensen and Thouless (1993) analysed the growth and the configurational stability of circular, buckling driven delamination by perturbing the front of the circular delamination, and then the driving force and toughness of the interface crack will be nonaxisymmetric naturally. Nilsson and Giannakopoulos (1995) used the finite element method to simulate the nonaxisymmetric growth process by perturbing the front of the circular delamination, or by perturbing the bi-axial load. In the results above, some nonaxisymmetric factors are introduced into the axisymmetric deformation in advance to analyse the nonaxisymmetric growth of circular delamination.

The axisymmetric growth of a circular debond, which is stable under certain conditions, is analysed in the paper by Zhang and Yu (1996a). The nonaxisymmetric secondary buckling which is bifurcated from the axisymmetric state is calculated by Zhang and Yu (1996b) by using the perturbation expansion method. The driving force and the mode-adjusted toughness of the interface crack vary along the periphery of the circular blister because of the nonaxisymmetric deformation, which is the mechanism of nonaxisymmetric growth of circular delamination.

This paper is organized as follows. First, the equations and boundary conditions which describe the buckling and growth of circular delamination in polar coordinates are deduced by recourse to the moving boundary variational principle developed by Chien (1980). In particular, the criterion for incipient advance of interface crack is obtained simultaneously. In Section 3.1, a high-order perturbation solution of axisymmetric buckling, which shows good agreement with the FEM results in the paper by Raju and Rao (1984), is derived. The critical loads corresponding to the nonaxisymmetric buckling mode are obtained in Section 3.2. Then the perturbative solutions of nonaxisymmetric deformation bifurcating from the axisymmetric buckling are obtained also. The axisymmetric growth of circular buckling-driven delamination is investigated in Section 4. Some new properties are revealed: under a certain residual compression the circular delaminations with the radii smaller than R_c will not buckle and the buckled delaminations with the radii smaller than R_g will not grow. With increasing of the pressure the buckled film changes from stationary into stable growth and instable growth. Finally, Section 5 contains a simulation of nonaxisymmetric growth of circular buckling-driven delamination. The driving force and the toughness of interface crack along the delamination front are calculated in Section 5.1. According to the critical condition $G = \Gamma(\psi)$, and without any biased assumptions about the delamination front shapes, the changing crack front corresponding to different nonaxisymmetric buckling modes ($n = 2, 3, 6, 8$) is simulated when some small nonaxisymmetric growth occurs. Simultaneously, the nonaxisymmetric growth process corresponding to the buckling mode $n = 2$ is simulated under a sequence of loads.

2. Statement of the problem

The circular delamination in the film/substrate is treated as a thin plate clamped to the substrate, as shown in Fig. 1. Both materials are assumed to be linear-elastic and isotropic, while the Young's

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Fig. 1. The delamination and the interface crack.

moduli for the film and the substrate, E_f and E_s, may be different, as may the Poisson's ratios, μ_f and μ_s. The substrate is modeled as being infinitely thick. The film thickness, t, is assumed to be far too small compared to the extent of the delamination R and the radius of the curvature of the crack front. This ensures that plane-strain conditions hold locally along the crack front. The stress state in the film in the unbuckled state is uniform, equi-biaxial compression $p(N/m)$. The deflection of the film following delamination is assumed to be of moderate size, which suggests framing the analysis within the classical Von Karman theory of moderate deflection of plate. So in polar coordinates system, the membrane and bending strains are defined as

$$\kappa_r = -\frac{\partial^2 w}{\partial r^2}, \quad \kappa_\theta = -\frac{1}{r}\frac{\partial w}{\partial r} - \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}, \quad \kappa_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)$$

$$\varepsilon_r = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2, \quad \varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)^2, \quad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}\right)$$

where u, v are in-plane displacements and w is normal deflection.

The strain energy of the delaminated portion of the film is

$$\Pi = \frac{D}{2}\iint_\Omega [\kappa_r^2 + \kappa_\theta^2 + 2\mu_f\kappa_r\kappa_\theta + 2(1-\mu_f)\kappa_{r\theta}^2]\,\mathrm{d}\Omega + \frac{C}{2}\iint_\Omega [\varepsilon_r^2 + \varepsilon_\theta^2 + 2\mu_f\varepsilon_r\varepsilon_\theta + 2(1-\mu_f)\varepsilon_{r\theta}^2]\,\mathrm{d}\Omega \tag{1}$$

where

$$D = \frac{E_f t^3}{12(1-\mu_f^2)}, \quad C = \frac{E_f t}{1-\mu_f^2}$$

are the bending and membrane stiffness of the film, which are defined in terms of the Young's modulus E_f and Poisson's ratio μ_f of the film. The region Ω is the delaminated portion. If the delamination spreads out along the interface, i.e., the integral region changes from Ω to $\Omega+\delta\Omega$, the variation of the strain energy is

$$
\begin{aligned}
\delta\Pi = {} & \frac{D}{2}\iint_{\Omega+\delta\Omega}[(\kappa_r+\delta\kappa_r)^2+(\kappa_\theta+\delta\kappa_\theta)^2+2\mu_f(\kappa_r+\delta\kappa_r)(\kappa_\theta+\delta\kappa_\theta)+2(1-\mu_f)(\kappa_{r\theta}+\delta\kappa_{r\theta})^2]\,\mathrm{d}\Omega \\
& +\frac{C}{2}\iint_{\Omega+\delta\Omega}[(\varepsilon_r+\delta\varepsilon_r)^2+(\varepsilon_\theta+\delta\varepsilon_\theta)^2+2\mu_f(\varepsilon_r+\delta\varepsilon_r)(\varepsilon_\theta+\delta\varepsilon_\theta)+2(1-\mu_f)(\varepsilon_{r\theta}+\delta\varepsilon_{r\theta})^2]\,\mathrm{d}\Omega \\
& +\oint_c \Gamma\delta n\cdot \mathrm{d}s-\Pi \\
= {} & D\iint_\Omega[\kappa_r\delta\kappa_r+\kappa_\theta\delta\kappa_\theta+\mu_f\kappa_r\delta\kappa_\theta+\mu_f\kappa_\theta\delta\kappa_r+2(1-\mu_f)\kappa_{r\theta}\delta\kappa_{r\theta}]\,\mathrm{d}\Omega \\
& +C\iint_\Omega[\varepsilon_r\delta\varepsilon_r+\varepsilon_\theta\delta\varepsilon_\theta+\mu_f\varepsilon_r\delta\varepsilon_\theta+\mu_f\varepsilon_\theta\delta\varepsilon_r+2(1-\mu_f)\varepsilon_{r\theta}\delta\varepsilon_{r\theta}]\,\mathrm{d}\Omega \\
& +\frac{D}{2}\iint_{\delta\Omega}[\kappa_r^2+\kappa_\theta^2+2\mu_f\kappa_r\kappa_\theta+2(1-\mu_f)\kappa_{r\theta}^2]\,\mathrm{d}\Omega \\
& +\frac{C}{2}\iint_{\delta\Omega}[\varepsilon_r^2+\varepsilon_\theta^2+2\mu_f\varepsilon_r\varepsilon_\theta+2(1-\mu_f)\varepsilon_{r\theta}^2]\,\mathrm{d}\Omega+\oint_c\Gamma\delta n\cdot\mathrm{d}s
\end{aligned}
\tag{2}
$$

where Γ is the surface energy per unit area, c the boundary of Ω with outward unit normal $\mathbf{n}$, δn the micro-increment of c along the normal direction, and ds the micro-increment along the tangent direction. In the region $\delta\Omega$ which is opened by δn, the integrand can be denoted by the value of a certain point on δn, and can be denoted by the value on c when $\delta n \to 0$. The error which is high-order of δn can be ignored in variation, so

$$
\begin{aligned}
& \frac{D}{2}\iint_{\delta\Omega}[\kappa_r^2+\kappa_\theta^2+2\mu_f\kappa_r\kappa_\theta+2(1-\mu_f)\kappa_{r\theta}^2]\,\mathrm{d}\Omega+\frac{C}{2}\iint_{\delta\Omega}[\varepsilon_r^2+\varepsilon_\theta^2+2\mu_f\varepsilon_r\varepsilon_\theta+2(1-\mu_f)\varepsilon_{r\theta}^2]\,\mathrm{d}\Omega \\
& =\frac{D}{2}\oint_c[\kappa_r^2+\kappa_\theta^2+2\mu_f\kappa_r\kappa_\theta+2(1-\mu_f)\kappa_{r\theta}^2]\cdot\delta n\cdot\mathrm{d}s+\frac{C}{2}\oint_c[\varepsilon_r^2+\varepsilon_\theta^2+2\mu_f\varepsilon_r\varepsilon_\theta+2(1-\mu_f)\varepsilon_{r\theta}^2]\cdot\delta n\cdot\mathrm{d}s
\end{aligned}
$$

while

$$
\begin{aligned}
& D\iint_\Omega[\kappa_r\delta\kappa_r+\kappa_\theta\delta\kappa_\theta+\mu_f\kappa_r\delta\kappa_\theta+\mu_f\kappa_\theta\delta\kappa_r+2(1-\mu_f)\kappa_{r\theta}\delta\kappa_{r\theta}]\,\mathrm{d}\Omega \\
& \quad +C\iint_\Omega[\varepsilon_r\delta\varepsilon_r+\varepsilon_\theta\delta\varepsilon_\theta+\mu_f\varepsilon_r\delta\varepsilon_\theta+\mu_f\varepsilon_\theta\delta\varepsilon_r+2(1-\mu_f)\varepsilon_{r\theta}\delta\varepsilon_{r\theta}]\,\mathrm{d}\Omega \\
& =\iint_\Omega\Bigg\{D\nabla^2\nabla^2 w-C\Bigg[\frac{\partial^2 w}{\partial r^2}(\varepsilon_r+\mu_f\varepsilon_\theta)+\frac{1}{r}\frac{\partial w}{\partial r}(\varepsilon_r+\mu_f\varepsilon_\theta)+\frac{\partial w}{\partial r}\frac{\partial}{\partial r}(\varepsilon_r+\mu_f\varepsilon_\theta)+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}(\varepsilon_\theta+\mu_f\varepsilon_r) \\
& \quad +\frac{1}{r^2}\frac{\partial w}{\partial\theta}(\varepsilon_\theta+\mu_f\varepsilon_r)+(1-\mu_f)\frac{1}{r}\frac{\partial w}{\partial\theta}\frac{\partial\varepsilon_{r\theta}}{\partial r}+(1-\mu_f)\frac{\partial w}{\partial r}\frac{\partial\varepsilon_{r\theta}}{r\,\partial\theta}+2(1-\mu_f)\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial\theta}\varepsilon_{r\theta}\Bigg]\Bigg\}\delta w\,\mathrm{d}\Omega
\end{aligned}
$$

$$-C\iint_{\Omega}\left[r\frac{\partial}{\partial r}(\varepsilon_r+\mu_f\varepsilon_\theta)+(\varepsilon_r+\mu_f\varepsilon_\theta)-(\varepsilon_\theta+\mu_f\varepsilon_r)+(1-\mu_f)\frac{\partial\varepsilon_{r\theta}}{\partial r}\right]\delta u\,\mathrm{d}\Omega$$

$$-C\iint_{\Omega}\left[\frac{\partial}{\partial\theta}(\varepsilon_\theta+\mu_f\varepsilon_r)+(1-\mu_f)\left(r\frac{\partial\varepsilon_{r\theta}}{\partial r}+2\varepsilon_{r\theta}\right)\right]\delta v\,\mathrm{d}\Omega$$

$$+D\oint_c\left[\frac{\partial^2 w}{\partial r^2}+\mu_f\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right]\delta\left(\frac{\partial w}{\partial r}\right)\bigg|_c r\,\mathrm{d}\theta$$

$$-C\oint_c(\varepsilon_r+\mu_f\varepsilon_\theta)\delta u|_c r\,\mathrm{d}\theta+C\oint_c\frac{1-\mu_f}{2}\varepsilon_{r\theta}\delta u|_c\,\mathrm{d}r+C\oint_c(\varepsilon_\theta+\mu_f\varepsilon_r)\delta v|_c\,\mathrm{d}r-C\oint_c\frac{1-\mu_f}{2}\varepsilon_{r\theta}\delta v|_c r\,\mathrm{d}\theta$$

where $\delta(\partial w/\partial r)|_c$ is the variation of $\partial w/\partial r$ on c when c is invariable. We define $\delta(\partial w/\partial r)|_c$ as the total variation of $\partial w/\partial r$ on c (i.e., considering the variation of c), then

$$\delta\left(\frac{\partial w}{\partial r}\bigg|\right)_c=\delta\left(\frac{\partial w}{\partial r}\right)\bigg|_c+\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial r}\right)\bigg|_c\cdot\delta n \tag{3}$$

or

$$\delta\left(\frac{\delta w}{\partial r}\right)\bigg|_c=\delta\left(\frac{\partial w}{\partial r}\bigg|\right)_c-\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial r}\right)\bigg|_c\cdot\delta n \tag{3'}$$

where $(\partial/\partial n)(\partial w/\partial r)|_c$ is the value of $(\partial/\partial n)(\partial w/\partial r)$ on c. For the same reason, we have

$$\delta u|_c=\delta(u|_c)-\frac{\partial u}{\partial n}\bigg|_c\cdot\delta n \tag{4}$$

$$\delta v|_c=\delta(v|_c)-\frac{\partial v}{\partial n}\bigg|_c\cdot\delta n \tag{5}$$

The meaning of each term is the same as that interpreted above.

Let α be the angle between the axis of $\theta=0$ and the normal direction $\mathbf{n}$ of a certain point on c, the differential of the polar axes can be written as

$$\mathrm{d}r=-\sin(\alpha-\theta)\cdot\mathrm{d}s,\quad r\,\mathrm{d}\theta=\cos(\alpha-\theta)\cdot\mathrm{d}s \tag{6}$$

For any function $A(r,\theta)$ which is sufficingly continuous for most mathematical purposes, the partial derivative is

$$\frac{\partial A}{\partial n}=\frac{\partial\alpha}{\partial r}\cos(\alpha-\theta)+\frac{1}{r}\frac{\partial A}{\partial\theta}\sin(\alpha-\theta) \tag{7}$$

Using the boundary conditions

$$w|_c = 0, \quad \left.\frac{\partial w}{\partial n}\right|_c = 0, \quad u|_c = 0, \quad v|_c = 0 \tag{8}$$

and (6), (7), the variation of the strain energy can be deduced as

$$\begin{aligned}\delta\Pi = \iint_\Omega \Bigg\{ & D\nabla^2\nabla^2 w - C\Bigg[\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)(\varepsilon_r + \mu_f\varepsilon_\theta) + \frac{\partial w}{\partial r}\frac{\partial}{\partial r}(\varepsilon_r + \mu_f\varepsilon_\theta) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}(\varepsilon_\theta + \mu_f\varepsilon_r) \\ & + \frac{1}{r^2}\frac{\partial w}{\partial \theta}(\varepsilon_\theta + \mu_f\varepsilon_r) + (1-\mu_f)\frac{1}{r}\frac{\partial w}{\partial \theta}\frac{\partial \varepsilon_{r\theta}}{\partial r} + (1-\mu_f)\frac{\partial w}{\partial r}\frac{\partial \varepsilon_{r\theta}}{r\,\partial \theta} + 2(1-\mu_f)\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial \theta}\varepsilon_{r\theta}\Bigg]\Bigg\}\delta w\,\mathrm{d}\Omega \\ & - C\iint_\Omega \left[r\frac{\partial}{\partial r}(\varepsilon_r + \mu_f\varepsilon_\theta) + (\varepsilon_r + \mu_f\varepsilon_\theta) - (\varepsilon_\theta + \mu_f\varepsilon_r) + (1-\mu_f)\frac{\partial \varepsilon_{r\theta}}{\partial r}\right]\delta u\,\mathrm{d}\Omega \\ & - C\iint_\Omega \left[\frac{\partial}{\partial \theta}(\varepsilon_\theta + \mu_f\varepsilon_r) + (1-\mu_f)\left(r\frac{\partial \varepsilon_{r\theta}}{\partial r} + 2\varepsilon_{r\theta}\right)\right]\delta v\,\mathrm{d}\Omega \\ & - \frac{C}{2}\oint_c \left[\left(\frac{\partial u}{\partial r}\right)^2 \cos 2(\alpha-\theta) + \frac{1-\mu_f}{2}\left(\frac{\partial v}{\partial r}\right)^2 \cos 2(\alpha-\theta) + \frac{1+\mu_f}{2}\frac{\partial u}{\partial r}\frac{\partial v}{\partial r}\sin 2(\alpha-\theta)\right]\delta n\,\mathrm{d}s \\ & - \frac{D}{2}\oint_c \left(\frac{\partial^2 w}{\partial r^2}\right)^2 \cos 2(\alpha-\theta)\delta n\,\mathrm{d}s + \oint_c \Gamma\cdot\delta n\,\mathrm{d}s\end{aligned} \tag{9}$$

where $\nabla^2(\cdot)$ is Laplace operator in polar coordinates. Because δw, δu, δv and δn are arbitrary, the Euler functions of this problem are

$$\begin{aligned}\nabla^2\nabla^2 w = \frac{C}{D}\Bigg[& \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)(\varepsilon_r + \mu_f\varepsilon_\theta) + \frac{\partial w}{\partial r}\frac{\partial}{\partial r}(\varepsilon_r + \mu_f\varepsilon_\theta) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}(\varepsilon_\theta + \mu_f\varepsilon_r) \\ & + \frac{1}{r^2}\frac{\partial w}{\partial \theta}(\varepsilon_\theta + \mu_f\varepsilon_r) + (1-\mu_f)\frac{1}{r}\frac{\partial w}{\partial \theta}\frac{\partial \varepsilon_{r\theta}}{\partial r} + (1-\mu_f)\frac{\partial w}{\partial r}\frac{\partial \varepsilon_{r\theta}}{r\,\partial \theta} + 2(1-\mu_f)\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial \theta}\varepsilon_{r\theta}\Bigg]\end{aligned} \tag{10}$$

$$r\frac{\partial}{\partial r}(\varepsilon_r + \mu_f\varepsilon_\theta) + (\varepsilon_r + \mu_f\varepsilon_\theta) - (\varepsilon_\theta + \mu_f\varepsilon_r) + (1-\mu_f)\frac{\partial \varepsilon_{r\theta}}{\partial r} = 0 \tag{11}$$

$$\frac{\partial}{\partial \theta}(\varepsilon_\theta + \mu_f\varepsilon_r) + (1-\mu_f)\left(r\frac{\partial \varepsilon_{r\theta}}{\partial r} + 2\varepsilon_{r\theta}\right) = 0 \tag{12}$$

and natural boundary condition

$$\frac{C}{2}\left[\left(\frac{\partial u}{\partial r}\right)^2\cos 2(\alpha-\theta)+\frac{1-\mu_f}{2}\left(\frac{\partial v}{\partial r}\right)^2\cos 2(\alpha-\theta)+\frac{1+\mu_f}{2}\frac{\partial u}{\partial r}\frac{\partial v}{\partial r}\sin 2(\alpha-\theta)\right]$$
$$+\frac{D}{2}\left(\frac{\partial^2 w}{\partial r^2}\right)^2\cos 2(\alpha-\theta)-\Gamma=0 \quad (13)$$

We define the energy release rate G as

$$G=\frac{C}{2}\left[\left(\frac{\partial u}{\partial r}\right)^2\cos 2(\alpha-\theta)+\frac{1-\mu_f}{2}\left(\frac{\partial v}{\partial r}\right)^2\cos 2(\alpha-\theta)+\frac{1+\mu_f}{2}\frac{\partial u}{\partial r}\frac{\partial v}{\partial r}\sin 2(\alpha-\theta)\right]$$
$$+\frac{D}{2}\left(\frac{\partial^2 w}{\partial r^2}\right)^2\cos 2(\alpha-\theta) \quad (14)$$

so (13) is the criterion for incipient advance of an interface crack.

Compared to the buckling theory of circular plate, the problem described by (8), (10)–(13) has an additional boundary condition (13) because of the undefined boundary. To express simply, eqns (10)–(14) can be formulated in terms of in-plane loads, with the results

$$D\cdot\nabla^2\nabla^2 w=\frac{\partial^2 w}{\partial r^2}N_r+\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)N_\theta$$
$$+2\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial\theta}-\frac{1}{r^2}\frac{\partial w}{\partial\theta}\right)N_{r\theta}-\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)p \quad (10')$$

$$\frac{\partial}{\partial r}(rN_r)-N_\theta+\frac{\partial N_{r\theta}}{\partial r}=0 \quad (11')$$

$$\frac{\partial N_\theta}{\partial\theta}+r\frac{\partial N_{r\theta}}{\partial r}+2N_{r\theta}=0 \quad (12')$$

$$G=\Gamma \quad (13')$$

and

$$G=\left[\frac{1}{2D}M^2+\frac{1}{2C}N_r^2+\frac{1}{2C(1-\mu_f)}N_{r\theta}^2\right]\cos 2(\alpha-\theta)-\frac{1}{2C}\frac{1+\mu}{1-\mu_f}N_rN_{r\theta}\sin 2(\alpha-\theta) \quad (14')$$

where M and N_r, N_θ, $N_{r\theta}$ are the bending moment and resultant stress–force per unit length caused by buckling, which can be written as

$$M=D(\partial^2 w/\partial r^2),\quad N_r=C(\varepsilon_r+\mu_f\varepsilon_\theta)-p,\quad N_\theta=C(\varepsilon_\theta+\mu_f\varepsilon_r)-p,\quad N_{r\theta}=C(1-\mu_f)\varepsilon_{r\theta}$$

3. Buckling and post-buckling calculation

We analyse the buckling and post-buckling of circular delamination firstly, not considering the growth, i.e., the boundary of the delamination remains circular. The analyses is similar to that given by Keller et al. (1962), Thompson and Hunt (1973), and Cheo and Reiss (1973, 1974) about the buckling and post-buckling of circular plate. The finite element method, finite difference method and perturbation method are used in these papers. But because of the complexity of the large deflection of the plate, these results for post-buckling of circular plate are incomplete, especially those of nonaxisymmetric secondary buckling, which bifurcates from the axisymmetric buckling, are much different. Further investigation of the axisymmetric and nonaxisymmetric buckling of circular delamination is developed in this paper.

3.1. Axisymmetric buckling of circular delamination

Under the axisymmetric condition, because every variable is irrelevant to θ and $v = 0$, the equations describing the buckling of circular delamination are simplified as

$$\dddot{w} + \frac{\ddot{w}}{r} - \frac{\dot{w}}{r^2} - \frac{12}{t^2}\left(\dot{u} + \frac{1}{2}\dot{w}^2 + \mu_f \frac{u}{r}\right)\dot{w} + \frac{P}{D}\dot{w} = 0 \tag{15}$$

$$\ddot{u} + \frac{\dot{u}}{r} - \frac{u}{r^2} + \dot{w}\ddot{w} + \frac{(1-\mu_f)}{2r}\dot{w}^2 = 0 \tag{16}$$

and boundary condition

$$w = 0, \quad \frac{\partial w}{\partial r} = 0, \quad u = 0 \tag{17}$$

We define dimensionless variables by

$$x = \frac{r}{R}, \quad \bar{w} = \sqrt{12(1-\mu_f^2)}\,\frac{w}{t}, \quad \bar{u} = \frac{uR}{t^2}, \quad \bar{P} = \frac{pR^2}{D} \tag{18}$$

and using the dimensionless center deflection s of the circular delamination as perturbative parameter, the perturbative expansions are given as

$$\bar{w} = w_1 s + w_3 s^3 + w_5 s^5 + w_7 s^7 \ldots \tag{19}$$

$$\bar{u} = u_2 s^2 + u_4 s^4 + u_6 s^6 + \cdots \tag{20}$$

$$\bar{P} = p_c + e_2^1 p_2 s^2 + e_4^1 p_4 s^4 + e_6^1 p_6 s^6 \ldots \tag{21}$$

Substituting (18)–(21) into (15)–(17), leads to the analytic solutions of w_1, u_2 as follows

$$\dot{w}_1 = A_1 J_1(\tau x) \tag{22}$$

$$u_2 = \frac{A_1^2}{4\tau}\{(1+\mu_f)J_0(\tau x)J_1(\tau x) + \mu_f \tau x[J_0^2(\tau) - J_0^2(\tau x) - J_1^2(\tau x)] - \tau x J_0^2(\tau)\} \tag{23}$$

where $\tau = \sqrt{p_c} = 3.8317$, $A_1 = -\tau/(1-J_0(\tau))$, J_0, J_1 are Bessel functions of zero and first-order, and $\tau = 3.8317$ is the smallest root of $J_1(x) = 0$. The numerical solutions of w_3, w_5, w_7, u_4 and u_6 can be obtained by using the shooting method. The eigenvalue p_c, p_2, p_4 and p_6 are obtained simultaneously, with the results

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Fig. 2. Buckling road comparing with FEM results reported by Raju and Rao (1984).

$$p_c = 14.6832, \quad p_2 = 1.410, \quad p_4 = 1.798\times 10^{-3}, \quad p_6 = 3.008\times 10^{-5}$$

The results are calculated with $\mu_f = 0.3$. The comparison between the perturbation solutions and the FEM results in the paper by Raju and Rao (1984) is shown in Fig. 2. Since only six separated solutions in the limits of $s \leqslant \sqrt{12(1-\mu_f^2)}$ are obtained by Raju and Rao (1984), we calculate the extended FEM results for $s > \sqrt{12(1-\mu_f^2)}$ from these six solutions. It can be seen from the figure that the two-order solution and the six-order solution coincide with the FEM results very well when p/p_c is small, while when p/p_c is larger than five, the two-order solution has a large deviation from the six-order solution and the FEM results. A further illustration on this point is given in Fig. 3. Figure 3a is the axisymmetric buckling morphologies calculated by the two-order solution. The deformation is concave in the center of the delamination when $p/p_c > 5$. This is not corresponding to the physical phenomenon. Figure 3b is the morphologies calculated by the six-order solution and it can be seen that the shape of the blisters are reasonable even when p/p_c is as large as 10.

3.2. Nonaxisymmetric buckling of circular delamination

As described by Cheo and Reiss (1974), a strip of large circumferential compressive stress develops adjacent to the edge of the plate (i.e., the delamination in this problem) with p increasing. The "width" of the strip decreases while the compressive stress intensity increases as p is increased. Thus, for sufficiently large p the strip may buckle unsymmetrically like a ring, in other words, the plate may buckle away from the axisymmetric buckled state by wrinkling near the edge into an unsymmetric state. To determine the wrinkling loads, we express nonaxisymmetric solutions w, u and v in the form

$$\bar{w}(x,\theta\,;P) = \bar{w}_0(x\,;P)+\bar{w}^*(x,\theta\,;P) \tag{24}$$

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Fig. 3. The axisymmetric buckling morphologies under a series of loads.

$$\bar{u}(x, \theta ; \bar{P}) = u_0(x ; \bar{P}) + u^*(x, \theta ; \bar{P}) \tag{25}$$

$$\bar{v}(x, \theta ; \bar{P}) = v^*(x, \theta ; \bar{P}) \tag{26}$$

where $\bar{w}$, $\bar{u}$ and $\bar{P}$ are dimensionless variables defined in (18) and $\bar{v} = v \cdot R/t^2$, w_0, u_0 are axisymmetric solutions obtained in Section 3.1, and w^*, u^* and v^* are the nonaxisymmetric deformation bifurcated from the axisymmetric buckling.

The dimensionless in-plane loads can be written as

$$\bar{N}_r = \frac{N_r R^2}{D}, \quad \bar{N}_\theta = \frac{N_\theta R^2}{D}, \quad \bar{N}_{r\theta} = \frac{N_{r\theta} R^2}{D} \tag{27}$$

We decompose $\bar{N}_r$, $\bar{N}_\theta$, $\bar{N}_{r\theta}$ into three parts

$$\bar{N}_r = N_r^0 + N_r^* - \bar{P}, \quad \bar{N}_\theta = N_\theta^0 + N_\theta^* - \bar{P}, \quad \bar{N}_{r\theta} = N_{r\theta}^* \tag{28}$$

where $\bar{P}$ is the compressive load, N_r^0, N_θ^0 denote the change from the plate state and N_r^*, N_θ^* are the change from the axisymmetric buckling state.

To calculate w^*, u^* and v^*, we insert (24), (25) and (26) into (8), (10′)–(12′) and obtain

$$\begin{aligned}\nabla^2\nabla^2 w^* = {} & \frac{\partial^2 w^*}{\partial x^2}(N_r^0 + N_r^*) + \frac{\partial^2 w^0}{\partial x^2} N_r^* + \left(\frac{1}{x}\frac{\partial w^*}{\partial x} + \frac{1}{x^2}\frac{\partial^2 w^*}{\partial \theta^2}\right)(N_\theta^0 + N_\theta^*) + \frac{1}{x}\frac{\partial w^0}{\partial x} N_\theta^* \\ & + 2\left(\frac{1}{x}\frac{\partial^2 w^*}{\partial x\,\partial\theta} - \frac{1}{x^2}\frac{\partial w^*}{\partial \theta}\right) N_{r\theta}^* - \bar{P}\left(\frac{\partial^2 w^*}{\partial x^2} + \frac{1}{x}\frac{\partial w^*}{\partial x} + \frac{1}{x^2}\frac{\partial^2 w^*}{\partial \theta^2}\right)\end{aligned} \tag{29}$$

$$\frac{\partial}{\partial x}(x N_r^\theta) - N_\theta^* + \frac{\partial N_{r\theta}^*}{\partial x} = 0 \tag{30}$$

$$\frac{\partial N_\theta^*}{\partial \theta} + x\frac{\partial N_{r\theta}^*}{\partial x} + 2N_{r\theta}^* = 0 \tag{31}$$

and boundary conditions

$$x = 1: \quad w^* = 0, \quad \frac{\partial w^*}{\partial x} = 0, \quad u^* = 0, \quad v^* = 0 \tag{32}$$

We seek the solutions of w^*, u^*, v^* and $\bar{P}$ in the form

$$w^* = w_1\varepsilon + w_2\varepsilon^2 + w_3\varepsilon^3 + \cdots \tag{33}$$

$$u^* = u_1\varepsilon + u_2\varepsilon^2 + u_3\varepsilon^3 + \cdots \tag{34}$$

$$v^* = v_1\varepsilon + v_2\varepsilon^2 + v_3\varepsilon^3 + \cdots \tag{35}$$

$$\bar{P} = p_0 + p_1\varepsilon + p_2\varepsilon^2 + \cdots \tag{36}$$

and expand the solutions w_0, u_0 of axisymmetric buckling in a power series in ε. The parameter ε is defined by

$$\varepsilon^2 = \iint_\Omega [(w - w_0)^2 + (u - u_0)^2 + v^2]\,\mathrm{d}\Omega = \iint_\Omega [(w^*)^2 + (u^*)^2 + (v^*)^2] x\,\mathrm{d}x\,\mathrm{d}\theta \tag{37}$$

Other definitions of ε can be used also.

In order to determine the coefficients w_i, u_i, v_i $(i = 1, 2, 3)$, we formulate these variables as follows

Table 1
The eigenvalue for different nonaxisymmetric buckling mode

n	1	2	3	4	5	6	7	8
p_0	136.66	107.85	113.0	129.27	150.31	174.77	201.44	231.04
p_0/p_c	9.31	7.35	7.70	8.81	10.24	11.90	13.72	15.74
p_2	—	0.089	1.768	5.366	6.731	32.98	105.57	131.57

* $p_c = 14.6832$ is the critical load for axisymmetric buckling.

$$\begin{aligned} w_1(x,\theta) &= w_{11}(x)(\sin n\theta+\cos n\theta) \\ u_1(x,\theta) &= u_{11}(x)(\sin n\theta+\cos n\theta) \\ v_1(x,\theta) &= v_{11}(x)(\cos n\theta-\sin n\theta) \end{aligned} \tag{38}$$

$$\begin{aligned} w_2(x,\theta) &= w_{20}(x)+w_{22}\sin 2n\theta \\ u_2(x,\theta) &= u_{20}(x)+u_{22}\sin 2n\theta \\ v_2(x,\theta) &= v_{20}(x)+v_{22}\sin 2n\theta \end{aligned} \tag{39}$$

$$\begin{aligned} w_3(x,\theta) &= w_{31}(x)(\sin n\theta+\cos n\theta+w_{33}(\sin^3 n\theta+\cos^3 n\theta) \\ u_3(x,\theta) &= u_{31}(x)(\sin n\theta+\cos n\theta)+u_{33}(\sin^3 n\theta+\cos^3 n\theta) \\ v_3(x,\theta) &= v_{31}(x)(\cos n\theta)-\sin n\theta)+v_{33}(\cos^3 n\theta-\sin^3 n\theta) \end{aligned} \tag{40}$$

Substituting (33)–(40) into (29)–(32), and equating coefficients of the same powers of ε, we get a series of linear ordinary differential equations, which are in terms of x, and the relevant boundary conditions. The numerical results of these equations which corresponds to n can be calculated by using the shooting method. The eigenvalues p_0, p_2 are shown in Table 1. In all cases that we studied, $p_1 \equiv 0$. So the nonaxisymmetric deformation of circular delamination under a certain compressive load P can be determined from (33)–(40). Figure 4 show the nonaxisymmetric out-of-plane deformation of the circular delamination in the nonaxisymmetric buckling mode $n = 2$, 3, 6, 8.

In order to certify our calculations of the non-axisymmetrical eigenvalue problems, we recalculated the non-axisymmetrical bifurcation by using the second order axisymmetrical solutions. A result ($n = 7, p_0 \approx 110.7$) which is similar with those by Cheo and Reiss (1974) was obtained. This result is much different from that in Table 1 because of the deference of the boundary condition. In our present paper the displacements are zero along the periphery, while in the paper of Cheo and Reiss the radial compressed load is a constant along the boundary.

4. Axisymmetric growth analysis

As mentioned by Argon et al. (1988, 1989) and Hutchinson et al. (1992), the initial flaws are small and tend to have smooth, nearly circular boundaries. So we analyse the axisymmetric growth of circular delamination first, and then simulate the nonaxisymmetric periphery of the blister.

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Fig. 4. The out-of-plane deformation of the circular delamination in different nonaxisymmetric buckling mode.

ss14024b

Fig. 4—*Continued.*

ss14025

Fig. 5. Dimensionless energy release rate for axisymmetric blister.

4.1. Driving force and mode-mixity of interface crack

In order to analyse the growth of the delamination, we must calculate the driving and resistant force of the crack first. The driving force (i.e., the energy release rate) of the crack along the front of the delamination can be evaluated from (14) or (14′) after the buckling deformation of the circular delamination is obtained. Under the condition of axisymmetric buckling, the crack mode is a combination of mode I and II, and (14′) can be rewritten as

$$G = \frac{1}{2D} M^2 + \frac{1}{2C} N_r^2 \tag{41}$$

The curve of G as a function of p/p_c is plotted in Fig. 5. The plot shows that the driving force G along the blister periphery increases monotonically with increasing p/p_c. The toughness of the interface crack can be written as (Hutchinson, et al., 1992)

$$\Gamma = \Gamma(\psi) = \Gamma_{1c} f(\psi), \quad f(\psi) = [1 + (\lambda - 1)\sin^2 \psi]^{-1} \tag{42}$$

where Γ_{1c} is the pure mode I interface toughness and the parameter $\lambda (0 \leqslant \lambda \leqslant 1)$ is chosen to fit experimental results for a given bimaterial interface. For $\lambda = 1$ it coincides with the classical mode-independent criterion $G = \Gamma_{1c}$, while for $\lambda < 1$, it is regarded as a phenomenological interface fracture criterion. In Jensen and Thouless (1993), $\lambda = 0.15$ gave good agreement with experimental results for a mica/aluminum system. ψ is the mixity of the interface crack, for the film/substrate systems, it can be written as

$$\psi = \frac{\sqrt{12} M \cos \omega + t N_r \sin \omega}{-\sqrt{12} M \sin \omega + t N_r \cos \omega} \tag{43}$$

ss14026

Fig. 6. Mode mixity parameter ψ vs p/p_c for three levels of elastic mismatch between film and substrate for axisymmetric blister.

where $\omega(\alpha, \beta)$ is a function tabulated in Suo and Hutchinson (1990), α and β, the two Dunders' elastic mismatch parameters, are defined as

$$\alpha = \frac{\bar{E}_f - \bar{E}_s}{\bar{E}_f + \bar{E}_s} \tag{44}$$

$$\beta = \frac{1}{2}\left[\frac{\bar{E}_f(1-2\mu_s)}{(1-\mu_s)(\bar{E}_f+\bar{E}_s)} - \frac{\bar{E}_s(1-2\mu_f)}{(1-\mu_f)(\bar{E}_f+\bar{E}_s)}\right] \tag{45}$$

Generally, α is more important in the two parameters for interfacial fracture problems. Moreover, a non-zero β-value complicates the application of interfacial fracture mechanics. In this paper, attention will be restricted to the mismatches with $\beta = 0$, either exactly or by approximation. Nothing of essence in the blister problem is lost by taking $\beta = 0$. The plots of mode mixedness ψ as a function of p/p_c for three levels of elastic mismatch ($\alpha = -0.8$, 0, 0.8) between film and substrate are shown in Fig. 6, and the toughness functions $f(\psi)$ are plotted in Fig. 7 with $\alpha = 0$ and $\mu_f = 0.3$. The toughness function $f(\psi)$ increases with increasing p/p_c (Fig. 7) because of the increase of $|\psi|$ (for p/p_c emerging from 1, ψ start at $\omega - \pi/2$ and reach peaks for $p/p_c \approx 8$). So it is possible that the axisymmetric growth of buckling driven delamination stops.

For discussing and presenting results simply, it is also useful to define a mode-adjusted crack driving force (Hutchinson et al., 1992)

$$F = G/f(\psi) \tag{46}$$

Thus, the condition for incipient fracture becomes

ss14027

Fig. 7. Family of interface toughness functions $f(\psi)$ vs p/p_c for axisymmetric blister with $\alpha = 0$ and $\mu_f = 0.3$.

$$F = \Gamma_{1c} \tag{47}$$

4.2. Spreading of circular blister

To analyse the axisymmetric growth of the blister, a load parameter $\tilde{p}$ is defined as

$$\tilde{p} = \frac{p}{E_f t} \tag{48}$$

The relationship between the dimensionless load $\bar{P}$ and the load parameter $\tilde{p}$ is

$$\bar{P} = 12(1-\mu_f^2)\left(\frac{R}{t}\right)^2 \cdot \tilde{p} \tag{49}$$

Equation (49) shows that under a certain load p (i.e. $\tilde{p}$), the dimensionless load $\bar{P}$ increases with the increase of R/t. That is, P increases with the growth of the delamination.

Under a certain residual compressive load p (i.e., p), there exists a critical radius R_c

$$R_c = \sqrt{\frac{14.6832}{12(1-\mu_f^2)\tilde{p}}} \cdot t \tag{50}$$

where t is the thickness of the film. R_c is defined by (50) such that the flat-state of the film will remain when the extent of the delamination R is smaller than R_c, and the film will buckle when $R \geqslant R_c$.

Referring to the dimension of the driving force G, Γ_{1c} can be represented by

ss14028

Fig. 8. Normalized mode-adjusted crack driving force for axisymmetric blister under three level of pressure with $\alpha = 0$ and $\mu_f = 0.3$.

$$\Gamma_{1c} = \Lambda \cdot \frac{D}{24} \cdot \frac{t^2}{R^4} = \Lambda \cdot \frac{E_f t}{288(1-\mu_f^2)} \cdot \left(\frac{t}{R}\right)^4 \tag{51}$$

where Λ is a constant. Equation (51) shows that Γ_{1c} is directly proportional to t under a certain extent of circular delamination R/t. If we choose mica for the film and aluminum for the substrate, some thickness-dependent Γ_{1c} are given by Hutchinson et al. (1992) as follows: $\Gamma_{1e} \approx 1.7$ Jm^{-2} for $t \approx 130$ μm; $\Gamma_{1c} \approx 0.8$ Jm^{-2} for $t \cong 60$ μm; and $\Gamma_{1c} \approx 0.4$ Jm^{-2} for $t \simeq 30$ μm. So $\Gamma_{1c}/t \approx 13{,}3333$ Jm^{-3} for this interface.

The conditions of stationary, stable growth and instable growth of the delamination with a certain interface toughness of $\Gamma_{1c}/t \approx 13{,}333$ Jm^{-3} for three levels of load are shown in Fig. 8. The extent-of the delamination is $R/t = 60$. The circular delamination buckles for $p = 5 \times 10^{-4}$ $E_f t$ (i.e., $P = 20.656$ when $R/t = 60$), but it would not extend out because $F/\Gamma_{1c} < 1.0$. The buckled delamination will spread under a larger load of $p = 8 \times 10^{-4}$ $E_f t$ ($P = 33.05$ when $R/t = 60$), and the growth will stop at point C because $\partial F/\partial(R/t) < 0$ and $F/\Gamma_{1c} = 1.0$. With the increasement of p, such as $p = 2 \times 10^3 E_f t$ ($P = 82.624$ when $R/t = 60$), the axisymmetric blister will spread unstably.

Under a certain residual compressive load p, the curve of mode-adjusted crack driving force F at the edge of axisymmetric circular blister as a function of R/t (i.e., P) is plotted in Fig. 9. There exist two characteristic radii R_c and R_g of the delamination, as shown in Fig. 9. The delaminated film will remain flat-state for $R < R_c$ and will buckle for $R \geqslant R_c$. The critical radius R_c is deduced from (50). The growing radius R_g varies with the difference of the interface toughness, so is the growth stability. The blister would spread without limit once the condition $F = \Gamma_{1cA}$ is first reached at its edge. In other words, a blister would either be sub-critical with $R < R_A$ or the film would completely delaminate. So in this case, $R_g = R_A$. With the increasing of the interface toughness, such as $\Gamma_{1c} = \Gamma_{1cB}$, the blister would propagate when $R = R_B$ (so $R_g = R_B$), but would stop at

ss14029

Fig. 9. Mode-adjusted crack driving force for axisymmetric blister under a certain pressure with $\alpha = 0$ and $\mu_f = 0.3$.

point C because $\Gamma_{1c} = \Gamma_{1cB}$ and $\partial F/\partial(R/t) < 0$. When $\Gamma_{1c} = \Gamma_{1cD}$, the blister would not spread in a large extent of delamination till the point E is reached. So $R_g = R_E$ in this condition.

5. Nonaxisymmetric growth simulation

5.1. *The driving force and the toughness of the interface crack*

If nonaxisymmetric buckling occurs, the interface crack mode changes to a combination of mode I, mode II and mode III. The energy release rate represented by (14) or (14′) shows this characteristic. The toughness of interface crack in the mixed mode of mode I and II has been analysed in detail and the acceptable results have been obtained. Concerning the combination of mode I and III, or mode II and III or even mode I, II and III, there are no perfect results up to now. If the nonaxisymmetric deformation is small and the growth of the crack is confined to the interface, we use the results of mode I and II mixed interface crack following the way suggested by Hutchinson et al. (1992), as expressed in (42). The mode mixedness ψ is calculated from (43). According to (46), we plot the curves of F/Γ_{1c} along the periphery of the blister when the pressure are large enough to cause nonaxisymmetric growth. Figure 10 illustrate the results for different nonaxisymmetric buckling mode $n = 2, 3, 6, 8$, where $\Gamma_{1c}/t \approx 13{,}333\ \mathrm{Jm}^{-3}$ is used.

5.2. *Nonaxisymmetric growth simulation*

We assume that the periphery of the delamination changes to be $r(\theta)$ after some small non-axisymmetric growths occur, so the slope of the normal direction $\mathbf{n}$ for a certain point on the boundary is

ss140210

Fig. 10. Normalized mode-adjusted crack driving force along the boundary of blister in different nonaxisymmetric buckling mode.

$$\tan\alpha = -\frac{(\mathrm{d}r/\mathrm{d}\theta)\cos\theta - r\sin\theta}{(\mathrm{d}r/\mathrm{d}\theta)\sin\theta + r\cos\theta} \tag{52}$$

it can be simplified as

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\tan(\alpha-\theta)\cdot r \tag{53}$$

Once the angle α of each point on the boundary c of the region of the delamination is calculated, the boundary configuration $r(\theta)$ can be determined easily by integrating (53). So we can simulate the nonaxisymmetric growth of the delamination by three steps:

1. To calculate the buckling deformation, as analysed in Section 3.
2. To determine the direction angle α of the normal line of the boundary c by means of the growth criterion of the interface crack, as illustrated in Section 5.1.
3. To simulate the boundary of the buckled delamination by integrating (53).

ss140211

Fig. 11. The boundary configuration of the blister in a series of nonaxisymmetric buckling modes.

The configurations of the blisters corresponding to different nonaxisymmetric buckling mode n $(=2, 3, 6, 8)$ are shown in Fig. 11. In addition, it can be seen from Table 1 that the smallest second critical load appears in the mode of $n = 2$. This means that the nonaxisymmetric buckling mode of $n = 2$ will take place firstly and the other nonaxisymmetric buckling modes will not occur if the circular delamination buckles and grows in the mode of $n = 2$. The plots in Fig. 12 interpret the process of nonaxisymmetric growth of the blister in the mode of $n = 2$ under a sequence of loads.

6. Conclusion

The axisymmetric and nonaxisymmetric buckling and growth of a circular delamination loaded in an equal bi-axial compression have been analysed in this paper. Some closed-form equations for the buckling and growth of the circular delamination are deduced by recourse to the moving boundary variational principle. The energy release rate of a mode I, II and III mixed interface crack are obtained simultaneously, and a relationship between the energy release rate and the boundary configuration is revealed, too.

ss140212

Fig. 12. The growth process of the blister in the nonaxisymmetric buckling mode $n = 2$ under a series of loads.

Two major features of axisymmetric growth of circular buckling-driven delamination have been explored:

1. A high-order perturbation solution of axisymmetric buckling, which shows good agreement with the FEM results, is obtained and applied to analyse the axisymmetric growth of buckling driven delamination. Some properties which differ from the results deduced from low-order expansions are revealed. The analyses in this paper indicate that there exist three conditions of stationary, stable growth and instable growth.
2. Two characteristic radii R_c and R_g exist under a certain residual pressure. The circular delaminated film will not buckle if its radius is less than R_c, and the buckled film will not spread out if $R < R_g$.

Two major results about the nonaxisymmetric growth are obtained in this paper:

3. The nonaxisymmetric buckling bifurcated from the axisymmetric buckling, which is considered as the mechanism of the nonaxisymmetric growth of buckling driven delamination, are calculated by using perturbation expansions.
4. Without any biased assumptions regarding the delamination front shapes, the nonaxisymmetric growths for different nonaxisymmetric buckling mode n $(=2, 3, 6, 8)$ are simulated. Under a sequence of loads, the growth process for nonaxisymmetric buckling mode $n = 2$ is obtained, too.

The approach developed in this paper can be used in the analysis of growth problems involving more complex shapes of delamination.

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